Wave equation
—wave mechanics of particles in curved space-time(Part I)

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In this paper, a universal wave equation of the motion of microscopic particles in curved space-time is established, which satisfies the conservation of probability. Under the classical limit conditions, the wave equation returns to the dynamic equation of special relativity. The application of the wave equation in the evolution of a flat expanding universe has preliminarily proved the correctness of the equation, and the new wave equation may open a window for us to understand physics within the Planck scale.

I. INTRODUCTION

General relativity and quantum mechanics are two epoch-making contributions of modern physics, they have achieved brilliant success in their respective fields. The successful field of general relativity is the movement of matter in macroscopic large-scale space-time, in this field, quantum effects can be ignored. The successful field of quantum mechanics is the microcosmic world of atoms and subatoms, in this field, gravity can be completely ignored. However, in the events of strong gravity and micro-scale space-time, both general relativity and quantum mechanics will be involved. In order to describe such events, it is necessary to combine general relativity with quantum mechanics [1].

How to combine general relativity with quantum mechanics? One of these attempts is to quantize the gravitational field, such as the canonical quantum theory of gravity [2]. However, the quantized gravitational field cannot be renormalized, which is a divergent theory. Another attempt is to quantize space and time itself [3], which is a new view, but lacks experimental support.

In this article, my goal is neither to quantize the gravitational field nor space-time, but to explore such a problem: in general relativity or curved space-time, what is the universal wave equation that the matter waves of microscopic particles satisfy? We require that the wave equation must meet the following conditions:

(1) Ensure that the statistical interpretation of wave function is valid. That is, the square of the amplitude of the wave function is proportional to the probability of finding the particle. In other words, the wave describing the particle is a probability wave.

(2) Guarantee the conservation of probability in curved space-time.

(3) Ensure that the superposition principle of states is still valid.

(4) The equation should contain the space-time metric tensor $g_{\mu\nu}$.

(5)The equation should contain three basic physical constants at the same time: gravitational constant G, Planck constant $\hbar$ and speed of light c, indicating that it is a wave equation that combines general relativity and quantum mechanics.

Obviously, neither Schrödinger equation nor Dirac equation can meet the above conditions at the same time. In this article, I will establish the wave equation of the motion of microscopic particles in the gravitational field(curved space-time), and lay the foundation for the further development of quantum mechanics.

II. ESTABLISHMENT OF WAVE EQUATION

My idea is to extend Feynman's path integral theory [4, 5] to curved space-time, and then derive the differential wave equation. Let $\psi(x, y, z, t)$ represent the probability amplitude of particle appearing at point (x, y, z) at time t, and $\psi(x^{'}, y^{'}, z^{'}, t^{'})$ represent the probability amplitude of particle appearing at point $(x^{'}, y^{'}, z^{'})$ at time $t^{'}$, then there is the following relationship

$$\psi(x, y, z, t) = \int K(x, y, z, t; x^{'}, y^{'}, z^{'}, t^{'})\psi(x^{'}, y^{'}, z^{'}, t^{'})dx^{'}dy^{'}dz^{'}, \tag{1}$$

where $K(x, y, z, t; x^{'}, y^{'}, z^{'}, t^{'})$ is called the propagator. According to Feynman's path integral theory [4, 5], the propagator of particle from point $A(x^{'}, y^{'}, z^{'})$ to point B(x, y, z) is

$$K(B, A) = C \sum_{all\ paths} \exp\{iS[r(t)]/\hbar\}, \tag{2}$$

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where

$$S[r(t)] = \int_{t_A}^{t_B} L(r, \dot{r}, t) dt \tag{3}$$

represents the action from point A to point B along the path r(t), L is the Lagrange of particle, C is the appropriate normalization constant. $\hbar$ in formula(2) is to make the phase factor dimensionless. The sum in formula(2) includes all possible paths from A to B.

In curved space-time, the wave function $\psi(x, y, z, t, \tau)$ is introduced to represent the probability amplitude of the particle appearing at space-time point (x, y, z, t) at time τ. τ is called proper time, (x, y, z, t) is the space-time coordinate, and there is a relationship between them $d\tau^2 = -ds^2 = -g_{\mu\nu}dx^\mu dx^\nu$, $g_{\mu\nu}$ is space-time metric tensor. By extending formula(1) to curved space-time, we get

$$\psi(x, y, z, t, \tau) = \int K(x, y, z, t, \tau; x^{'}, y^{'}, z^{'}, t^{'}, \tau^{'})\psi(x^{'}, y^{'}, z^{'}, t^{'}, \tau^{'})\sqrt{-g}dt^{'}dx^{'}dy^{'}dz^{'}, \tag{4}$$

where $g = |g_{\mu\nu}|$ is the determinant of the metric tensor, and $\sqrt{-g}d^4x^{'}$ is the invariant volume element. $K(x, y, z, t, \tau; x^{'}, y^{'}, z^{'}, t^{'}, \tau^{'})$ is the propagator in curved space-time, how to calculate it? I also extend Feynman path integral formula(2) to curved space-time. Let the propagator of particle from point $A(x^{'}, y^{'}, z^{'}, t^{'})$ to point B(x, y, z, t) be

$$K(B, A) = C \sum_{all\ paths} \exp\left\{\frac{iS[r(\tau)]}{2l_P}\right\}, \tag{5}$$

where

$$S[r(\tau)] = \int_{\tau_A}^{\tau_B} d\tau = \int_{\tau_A}^{\tau_B} \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}d\tau \tag{6}$$

represents the action from point A to point B along the path $r(\tau)$, $L \equiv \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}$ is the Lagrange function of the particle, C is a appropriate normalization constant. $l_P = (G\hbar/c^3)^{1/2} = 1.61 \times 10^{-33}$cm, which is the Planck length. After introducing Planck length l_P, the phase factor in formula(5) is dimensionless. Like formula(2), the sum in formula(5) includes all possible paths from A to B.

Formulas (4) and (5) are the basic formulas that Feynman's path integral theory is extended to curved space-time. From them, the differential wave equation of particle can be derived.

We consider the state of particle $\psi(x, y, z, t, \tau + \varepsilon)$ at time $\tau + \varepsilon$, which has the following relationship with the state of particle $\psi(x^{'}, y^{'}, z^{'}, t^{'}, \tau)$ at time τ

$$\psi(x, y, z, t, \tau + \varepsilon) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K(x, y, z, t, \tau + \varepsilon; x^{'}, y^{'}, z^{'}, t^{'}, \tau)\psi(x^{'}, y^{'}, z^{'}, t^{'}, \tau)\sqrt{-g}dt^{'}dx^{'}dy^{'}dz^{'}. \tag{7}$$

When $\varepsilon \to 0$, in this infinitesimal proper time interval, according to formula(5), the propagator can be expressed as

$$K(x, y, z, t, \tau + \varepsilon; x^{'}, y^{'}, z^{'}, t^{'}, \tau) = C \exp\left[\frac{i\varepsilon}{2l_P}\right], \tag{8}$$

where C is a constant to be determined. Substituting formula(8) into formula(7), we obtain

$$\psi(x, y, z, t, \tau + \varepsilon) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C \exp\left[\frac{i\varepsilon}{2l_P}\right]\psi(x^{'}, y^{'}, z^{'}, t^{'}, \tau)\sqrt{-g}(x^{'}, y^{'}, z^{'}, t^{'})dt^{'}dx^{'}dy^{'}dz^{'}. \tag{9}$$

Let $t^{'} = t + \eta_0, x^{'} = x + \eta_1, y^{'} = y + \eta_2, z^{'} = z + \eta_3$, then formula(9) can be written as

$$\begin{aligned}\psi(x, y, z, t, \tau + \varepsilon) = &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C \exp\left[\frac{i\varepsilon}{2l_P}\right]\psi(x + \eta_1, y + \eta_2, z + \eta_3, t + \eta_0, \tau) \\ &\cdot \sqrt{-g}(x + \eta_1, y + \eta_2, z + \eta_3, t + \eta_0)d\eta_0 d\eta_1 d\eta_2 d\eta_3.\end{aligned} \tag{10}$$

When ε tends to zero, it can be seen from the constraint condition $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$ that the main contribution of the integral comes from the region $\eta_{0,1,2,3} \approx 0$. Taking the integrand function in the above formula as the Taylor expansion of

$\eta_0, \eta_1, \eta_2, \eta_3$, we obtain

$$
\begin{aligned}
\psi(x,y,z,t,\tau)+\varepsilon\frac{\partial\psi}{\partial\tau}=&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}C\left[1+\frac{i\varepsilon}{2l_P}\right]\left[\psi(x,y,z,t,\tau)+\eta_1\frac{\partial\psi}{\partial x}+\eta_2\frac{\partial\psi}{\partial y}+\eta_3\frac{\partial\psi}{\partial z}+\eta_0\frac{\partial\psi}{\partial t}\right.\\
&+\frac{1}{2}\eta_1^2\frac{\partial^2\psi}{\partial x^2}+\frac{1}{2}\eta_2^2\frac{\partial^2\psi}{\partial y^2}+\frac{1}{2}\eta_3^2\frac{\partial^2\psi}{\partial z^2}+\frac{1}{2}\eta_0^2\frac{\partial^2\psi}{\partial t^2}+\eta_1\eta_2\frac{\partial^2\psi}{\partial x\partial y}+\eta_1\eta_3\frac{\partial^2\psi}{\partial x\partial z}+\eta_1\eta_0\frac{\partial^2\psi}{\partial x\partial t}\\
&\left.+\eta_2\eta_3\frac{\partial^2\psi}{\partial y\partial z}+\eta_2\eta_0\frac{\partial^2\psi}{\partial y\partial t}+\eta_3\eta_0\frac{\partial^2\psi}{\partial z\partial t}+\cdots\right]\\
&\cdot\left[\sqrt{-g}(x,y,z,t)+\eta_1\frac{\partial\sqrt{-g}}{\partial x}+\eta_2\frac{\partial\sqrt{-g}}{\partial y}+\eta_3\frac{\partial\sqrt{-g}}{\partial z}+\eta_0\frac{\partial\sqrt{-g}}{\partial t}\right.\\
&+\frac{1}{2}\eta_1^2\frac{\partial^2\sqrt{-g}}{\partial x^2}+\frac{1}{2}\eta_2^2\frac{\partial^2\sqrt{-g}}{\partial y^2}+\frac{1}{2}\eta_3^2\frac{\partial^2\sqrt{-g}}{\partial z^2}+\frac{1}{2}\eta_0^2\frac{\partial^2\sqrt{-g}}{\partial t^2}+\eta_1\eta_2\frac{\partial^2\sqrt{-g}}{\partial x\partial y}+\eta_1\eta_3\frac{\partial^2\sqrt{-g}}{\partial x\partial z}+\eta_1\eta_0\frac{\partial^2\sqrt{-g}}{\partial x\partial t}\\
&\left.+\eta_2\eta_3\frac{\partial^2\sqrt{-g}}{\partial y\partial z}+\eta_2\eta_0\frac{\partial^2\sqrt{-g}}{\partial y\partial t}+\eta_3\eta_0\frac{\partial^2\sqrt{-g}}{\partial z\partial t}+\cdots\right]d\eta_0 d\eta_1 d\eta_2 d\eta_3.
\end{aligned}
\tag{11}
$$

In the $\varepsilon \to 0$, $\eta_0, \eta_1, \eta_2, \eta_3 \to 0$ limit, we ignore all infinitesimal terms of higher order and equation(11) can be recombined as

$$
\begin{aligned}
\psi(x,y,z,t,\tau)+\varepsilon\frac{\partial\psi}{\partial\tau}=&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}C\left\{\psi(x,y,z,t,\tau)\sqrt{-g}(x,y,z,t)+\frac{1}{2}\eta_1^2\frac{\partial^2}{\partial x^2}\left[\sqrt{-g}\psi\right]\right.\\
&+\frac{1}{2}\eta_2^2\frac{\partial^2}{\partial y^2}\left[\sqrt{-g}\psi\right]+\frac{1}{2}\eta_3^2\frac{\partial^2}{\partial z^2}\left[\sqrt{-g}\psi\right]+\frac{1}{2}\eta_0^2\frac{\partial^2}{\partial t^2}\left[\sqrt{-g}\psi\right]\\
&+[\cdots]\eta_0+[\cdots]\eta_1+[\cdots]\eta_2+[\cdots]\eta_3+[\cdots]\eta_0\eta_1+[\cdots]\eta_0\eta_2+[\cdots]\eta_0\eta_3+[\cdots]\eta_1\eta_2\\
&\left.+[\cdots]\eta_1\eta_3+[\cdots]\,\eta_2\eta_3+\frac{i\varepsilon}{2l_P}\psi(x,y,z,t,\tau)\sqrt{-g}(x,y,z,t)\right\}d\eta_0 d\eta_1 d\eta_2 d\eta_3.
\end{aligned}
\tag{12}
$$

Formula(12) holds, then

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}C\sqrt{-g}(x,y,z,t)d\eta_0 d\eta_1 d\eta_2 d\eta_3=1 \tag{13}$$

is required.

We have known that the integral formula

$$\int_{-\infty}^{+\infty}\exp\left[\frac{i\eta_0^2}{2l_P\varepsilon}\right]d\eta_0\int_{-\infty}^{+\infty}\exp\left[\frac{i\eta_1^2}{2l_P\varepsilon}\right]d\eta_1\int_{-\infty}^{+\infty}\exp\left[\frac{i\eta_2^2}{2l_P\varepsilon}\right]d\eta_2\int_{-\infty}^{+\infty}\exp\left[\frac{i\eta_3^2}{2l_P\varepsilon}\right]d\eta_3=(2\pi l_P i\varepsilon)^2, \tag{14}$$

so we can take the undetermined constant C as

$$C=\frac{1}{\sqrt{-g}}\frac{1}{(2\pi l_P i\varepsilon)^2}\exp\left[\frac{i\eta_0^2}{2l_P\varepsilon}\right]\exp\left[\frac{i\eta_1^2}{2l_P\varepsilon}\right]\exp\left[\frac{i\eta_2^2}{2l_P\varepsilon}\right]\exp\left[\frac{i\eta_3^2}{2l_P\varepsilon}\right]. \tag{15}$$

We then use the integral formula

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}C\eta_\mu d\eta_0 d\eta_1 d\eta_2 d\eta_3=0,\quad \mu=0,1,2,3. \tag{16}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}C\eta_\mu\eta_\nu d\eta_0 d\eta_1 d\eta_2 d\eta_3=0,\quad \mu\neq\nu \tag{17}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}C\eta_\mu^2 d\eta_0 d\eta_1 d\eta_2 d\eta_3=\frac{il_P\varepsilon}{\sqrt{-g}}, \tag{18}$$

the equation(12) is converted into

$$\psi(x,y,z,t,\tau)+\varepsilon\frac{\partial\psi}{\partial\tau}=\psi(x,y,z,t,\tau)+\frac{il_P\varepsilon}{2\sqrt{-g}}\left[\frac{\partial^2}{\partial x^2}(\sqrt{-g}\psi)+\frac{\partial^2}{\partial y^2}(\sqrt{-g}\psi)\right.$$
$$\left.+\frac{\partial^2}{\partial z^2}(\sqrt{-g}\psi)+\frac{\partial^2}{\partial t^2}(\sqrt{-g}\psi)\right]+\frac{i\varepsilon}{2l_P}\psi(x,y,z,t,\tau), \tag{19}$$

namely

$$il_P\frac{\partial\psi}{\partial\tau}=-\frac{l_P^2}{2\sqrt{-g}}\left[\frac{\partial^2}{\partial x^2}(\sqrt{-g}\psi)+\frac{\partial^2}{\partial y^2}(\sqrt{-g}\psi)\right.$$
$$\left.+\frac{\partial^2}{\partial z^2}(\sqrt{-g}\psi)+\frac{\partial^2}{\partial t^2}(\sqrt{-g}\psi)\right]-\frac{1}{2}\psi(x,y,z,t,\tau), \tag{20}$$

which is the differential wave equation of the motion of microscopic particles in curved space-time(gravitational field).

(1) This is a complex equation, and the statistical interpretation of the wave function is still valid in the curved space. For the normalized wave function, which satisfies

$$\int|\psi(x,y,z,t,\tau)|^2\sqrt{-g}d^4x=1, \tag{21}$$

$\rho=|\psi|^2$ represents the probability density, and which conforms to the probability conservation equation(I will prove it in the next section).

(2) Wave equation(20) is linear, that is, if ψ_1 and ψ_2 are both solutions of the equation, then the linear superposition of ψ_1 and ψ_2, $a\psi_1+b\psi_2$, is also the solution of the equation, which conforms to the superposition principle of states in quantum mechanics.

(3) In the natural unit system, $l_P=t_Pc=t_P$, the Planck time $t_P=5.38\times10^{-44}s$, so another form of wave equation is to replace l_P in equation(20) with t_P. Equation(20), which contains three basic physical constants, gravitational constant G, Planck constant $\hbar$ and speed of light c (because $l_P=(G\hbar/c^3)^{1/2}$), is wave equation of particle motion in general relativity. If Dirac equation combines special relativity with quantum mechanics, wave equation(20) is the combination of general relativity and quantum mechanics.

III. CONSERVATION OF PROBABILITY AND CLASSICAL LIMIT OF WAVE EQUATION

We define the four-dimensional operator $\square$

$$\square=\hat{e_x}\frac{\partial}{\partial x}+\hat{e_y}\frac{\partial}{\partial y}+\hat{e_z}\frac{\partial}{\partial z}+\hat{e_t}\frac{\partial}{\partial t}, \tag{22}$$

where $\hat{e_x},\hat{e_y},\hat{e_z},\hat{e_t}$ is the unit vector of four-dimensional rectangular coordinates, then

$$\square\cdot\square=\square^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial t^2}, \tag{23}$$

so wave equation(20) can be rewritten as

$$il_P\frac{\partial\psi}{\partial\tau}=-\frac{l_P^2}{2\sqrt{-g}}\square^2(\sqrt{-g}\psi)-\frac{1}{2}\psi. \tag{24}$$

By separating the amplitude and phase of the wave function, we let

$$\psi=Re^{\frac{iS}{l_P}}, \tag{25}$$

where R and S are real numbers. Substituting formula(25) into equation(24), we get

$$\frac{\partial R}{\partial\tau}=-\frac{1}{2}\left(R\square^2S+2\square R\cdot\square S+\frac{2R}{\sqrt{-g}}\square\sqrt{-g}\cdot\square S\right), \tag{26}$$

$$\frac{\partial S}{\partial \tau}=-\frac{1}{2}(\Box S)^2+\frac{1}{2}+\frac{l_P^2}{2}\frac{\Box^2 R}{R}+\frac{l_P^2}{2\sqrt{-g}}\Box^2\sqrt{-g}+\frac{l_P^2}{\sqrt{-g}R}\Box\sqrt{-g}\cdot\Box R. \tag{27}$$

Equations(26) and (27) are completely equivalent to wave equation(24), now we will discuss their physical significance respectively.

I first prove that equation(26) is the differential expression of the conservation of probability in curved space-time. The probability density ρ and flow density J are defined as

$$\rho=|\psi|^2=R^2, \tag{28}$$

$$J=\frac{il_P}{2}[\psi\Box\psi^*-\psi^*\Box\psi]. \tag{29}$$

Substituting formula(25) into equation(29), we get

$$J=R^2\Box S. \tag{30}$$

Both sides of equation(26) are multiplied by $2R$, one can obtain

$$\frac{\partial \rho}{\partial \tau}+\Box\cdot J+\frac{2J}{\sqrt{-g}}\cdot\Box\sqrt{-g}=0. \tag{31}$$

The definition of four-dimensional speed is

$$U^\mu=\frac{dx^\mu}{d\tau}. \tag{32}$$

From formula(30), the four-dimensional speed can also be expressed as

$$U^\mu=\frac{J^\mu}{\rho}=(\Box S)^\mu. \tag{33}$$

Equation(31) can be rewritten as

$$\frac{\partial \rho}{\partial \tau}+\frac{\partial J^\mu}{\partial x^\mu}+\frac{2J^\mu}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\sqrt{-g})=0 \tag{34}$$

By integrating(34) with any volume element in curved space-time, we have

$$\int\frac{\partial \rho}{\partial \tau}\sqrt{-g}d^4x+\int\frac{\partial J^\mu}{\partial x^\mu}\sqrt{-g}d^4x+\int\frac{2J^\mu}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\sqrt{-g})\sqrt{-g}d^4x=0. \tag{35}$$

The first term in formula(35) can be expressed as

$$\int\frac{\partial \rho}{\partial \tau}\sqrt{-g}d^4x=\frac{\partial}{\partial \tau}\left[\int\rho\sqrt{-g}d^4x\right]-\int\rho\frac{\partial\sqrt{-g}}{\partial \tau}d^4x, \tag{36}$$

and the second and third terms in formula(35) can be combined as

$$\begin{aligned}&\int\left(\frac{\partial J^\mu}{\partial x^\mu}+\frac{2J^\mu}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\sqrt{-g})\right)\sqrt{-g}d^4x\\&=\int\left[\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}J^\mu\right)+\frac{J^\mu}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\sqrt{-g})\right]\sqrt{-g}d^4x\\&=\int J^\mu_{;\mu}\sqrt{-g}d^4x+\int\frac{J^\mu}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\sqrt{-g})\sqrt{-g}d^4x,\end{aligned} \tag{37}$$

where

$$J^\mu_{;\mu}=\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}J^\mu\right) \tag{38}$$

which is the covariant divergence of the four-dimensional probability flow density J^μ.

From formulas(36) and (37), equation(35) can be written as

$$\frac{\partial}{\partial\tau}\left[\int\rho\sqrt{-g}d^4x\right]-\int\rho\frac{\partial\sqrt{-g}}{\partial\tau}d^4x+\int J^\mu_{;\mu}\sqrt{-g}d^4x+\int\frac{J^\mu}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\sqrt{-g})\sqrt{-g}d^4x=0 \tag{39}$$

The second and fourth items in the above formula are combined as

$$\begin{aligned}&\int\left(J^\mu\frac{\partial\sqrt{-g}}{\partial x^\mu}-\rho\frac{\partial\sqrt{-g}}{\partial\tau}\right)d^4x\\&=\int\rho\left[\frac{\partial\sqrt{-g}}{\partial x^\mu}U^\mu-\frac{\partial\sqrt{-g}}{\partial\tau}\right]d^4x\\&=\int\rho\left[\frac{\partial\sqrt{-g}}{\partial x^\mu}\frac{dx^\mu}{d\tau}-\frac{\partial\sqrt{-g}}{\partial\tau}\right]d^4x\\&=0\end{aligned} \tag{40}$$

The reason for the last step in the derivation of the above formula is $\sqrt{-g}=\sqrt{-g}(x(\tau),y(\tau),z(\tau),t(\tau))$, and we have

$$\begin{aligned}\frac{\partial\sqrt{-g}}{\partial\tau}&=\frac{\partial\sqrt{-g}}{\partial x}\frac{dx}{d\tau}+\frac{\partial\sqrt{-g}}{\partial y}\frac{dy}{d\tau}+\frac{\partial\sqrt{-g}}{\partial z}\frac{dz}{d\tau}+\frac{\partial\sqrt{-g}}{\partial t}\frac{dt}{d\tau}\\&=\frac{\partial\sqrt{-g}}{\partial x^\mu}\frac{dx^\mu}{d\tau}\end{aligned} \tag{41}$$

So formula(39) is finally converted to

$$\frac{\partial}{\partial\tau}\left[\int\rho\sqrt{-g}d^4x\right]+\int J^\mu_{;\mu}\sqrt{-g}d^4x=0 \tag{42}$$

Using Gauss theorem in four-dimensional curved space-time, formula(42) can be written as

$$\frac{\partial}{\partial\tau}\left[\int\rho\sqrt{-g}d^4x\right]+\oint_{s_3}\sqrt{-g}J^\mu d^3S_\mu=0, \tag{43}$$

which is exactly the conservation equation of probability in curved space-time.

Now let's discuss the classical limit of equation(27). Under the classical limit condition, Planck length $l_P\to 0$, the l_P^2 terms in equation(27) can be ignored, then equation(27) can be simplified as

$$\frac{\partial S}{\partial\tau}+\frac{1}{2}(\Box S)^2-\frac{1}{2}=0. \tag{44}$$

Because four-dimensional velocity$\mathbf{U}=\Box S$, equation(44) can be rewritten as

$$\frac{\partial S}{\partial\tau}+\frac{1}{2}\mathbf{U}^2-\frac{1}{2}=0. \tag{45}$$

Taking four-dimensional gradient on both sides of equation(45), and using $\mathbf{U}=\Box S$, we get

$$\frac{\partial\mathbf{U}}{\partial\tau}+(\mathbf{U}\cdot\Box)\,\mathbf{U}=0. \tag{46}$$

Because the four-dimensional velocity can also be written as $\mathbf{U}=\mathbf{U}(x(\tau),y(\tau),z(\tau),t(\tau),\tau)$, we have

$$\begin{aligned}\frac{d\mathbf{U}}{d\tau}&=\frac{\partial\mathbf{U}}{\partial\tau}+\frac{\partial\mathbf{U}}{\partial x}\frac{dx}{d\tau}+\frac{\partial\mathbf{U}}{\partial y}\frac{dy}{d\tau}+\frac{\partial\mathbf{U}}{\partial z}\frac{dz}{d\tau}+\frac{\partial\mathbf{U}}{\partial t}\frac{dt}{d\tau}\\&=\frac{\partial\mathbf{U}}{\partial\tau}+(\mathbf{U}\cdot\Box)\,\mathbf{U}\end{aligned} \tag{47}$$

At last, equation(46) can be expressed as

$$\frac{d\mathbf{U}}{d\tau}=0, \tag{48}$$

which is exactly the equation of motion of free particles in special relativity. This means that when the planck length l_P approaches zero, the gravitational and quantum effects disappear simultaneously, and the wave equation returns to the classical dynamics equation of special relativity.

IV. THE EVOLUTION OF A FLAT EXPANDING UNIVERSE —A SIMPLE APPLICATION OF THE WAVE EQUATION

Wave equation(20) is a complicated partial differential complex equation, which is difficult to solve in general, but can be solved in special cases. In this section, I will use the wave equation to discuss the dynamic evolution of the flat expanding universe.

In a flat, expanding universe, the metric tensor is [6]

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix} \tag{49}$$

which is called the Robertson-Walker(RW) metric, and $a(t)$ is the scale factor of the expanding universe.

Substituting RW metric into Einstein field equation, one can get

$$3\ddot{a} = -4\pi G(\rho + 3p)a, \tag{50}$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2. \tag{51}$$

ρ and p are the density and pressure of the medium, which are uniform in space and only change with time t, i. e. $\rho = \rho(t)$, $p = p(t)$. Equations(50) and (51) are called Friedmann equations, which are the basic equations of evolution of the universe. So we have two independent equations(50) and (51), but three unknown functions $\rho(t)$, $p(t)$ and $a(t)$, thus they are incomplete. In the traditional discussions of the evolution of the universe, we need to artificially add a medium's equation of state $p = p(\rho)$ to form a complete set of dynamic equations. In a complete theory, the equation of state should be the natural result of the theory, so the theory which only contains Friedmann equations is incomplete. Now, we have an independent motion equation(20), because the wave equation is only applicable to the motion of non-zero mass matter, for the evolution stage of the universe dominated by non-zero mass matter, the Friedmann equations and the new equation of motion(20) together can form a complete equation set.

In comoving coordinate system, $d\tau = dt$. To separate the amplitude and phase of the wave function, we define

$$\psi(t) = R(t)\mathrm{e}^{\frac{iS(t)}{l_P}}. \tag{52}$$

Substituting RW metric into equations of motion(26) and (27), we obtain

$$\dot{R} = -\frac{1}{2}\left[R\ddot{S} + 2\dot{R}\dot{S} + \frac{6R}{a}\dot{a}\dot{S}\right] \tag{53}$$

$$\dot{S} = -\frac{1}{2}\dot{S}^2 + \frac{1}{2} + \frac{l_P^2}{2}\frac{\ddot{R}}{R} + \frac{3l_P^2}{a^2}\dot{a}^2 + \frac{3l_P^2}{2a}\ddot{a} + \frac{3l_P^2}{aR}\dot{a}\dot{R} \tag{54}$$

Therefore we have a set of simultaneous equations composed of four independent equations(50), (51), (53) and (54).

$$\begin{cases} \ddot{a} = -\dfrac{4\pi G}{3}(\rho + 3p)a \\ \dot{a}^2 = \dfrac{8\pi G}{3}\rho a^2 \\ \dot{R} = -\dfrac{1}{2}\left[R\ddot{S} + 2\dot{R}\dot{S} + \dfrac{6R}{a}\dot{a}\dot{S}\right] \\ \dot{S} = -\dfrac{1}{2}\dot{S}^2 + \dfrac{1}{2} + \dfrac{l_P^2}{2}\dfrac{\ddot{R}}{R} + \dfrac{3l_P^2}{a^2}\dot{a}^2 + \dfrac{3l_P^2}{2a}\ddot{a} + \dfrac{3l_P^2}{aR}\dot{a}\dot{R} \end{cases} \tag{55}$$

It is seemingly that we have five unknown functions $\rho(t)$, $p(t)$, $a(t)$, $R(t)$ and $S(t)$, only four independent equations, which are still incomplete. However, considering the statistical interpretation of wave function $\rho(t) = |\psi|^2 = R^2(t)$, there are only four independent unknown functions, so the set of equations (55) is complete and can be solved.

The exact solution to equation system(55) is

$$
\begin{cases}
\rho(t) = \left(\dfrac{1}{\sqrt{6\pi G}t + C_1}\right)^2 \\
p(t) = 0 \\
a(t) = C_2\left(\sqrt{6\pi G}t + C_1\right)^{2/3} \\
R(t) = \dfrac{1}{\sqrt{6\pi G}t + C_1} \\
S(t) = t + C_3
\end{cases}
\tag{56}
$$

where C_1, C_2 and C_3 are integral constants, which are determined by initial conditions. From the solution of the equations, we can see that the equation of state $p = 0$ and ρa^3 =Const. are both natural corollaries of the complete dynamic evolution equation set(55). In addition, the four-dimensional velocity $U^0 = \dot{S} = 1$ and $\dot{S}^2 = U^0 U_0 = -1$ are also the natural results of the equations.

V. EQUATIONS OF SPACE-TIME STRUCTURE AND PARTICLE MOTION

In general relativity, particles move along the geodesic line in curved spacetime, and metric tensor $g_{\mu\nu}$ in the geodesic line is obtained by solving the Einstein field equation. So in general, we first assume that the distribution of matter is known (that is, we have known the energy momentum tensor $T_{\mu\nu}$), and then calculate the space-time metric tensor $g_{\mu\nu}$ from Einstein field equation, and then calculate the motion of particles from the geodesic equation. Because the motion of particles is also a part of the energy momentum tensor, through the field equation the motion of particles will affect the structure of space-time $g_{\mu\nu}$, which in turn will affect the motion of particles again. In the most general case, the distribution of matter $T_{\mu\nu}$ and the space-time structure $g_{\mu\nu}$ should be both unknown, in this case, how can we calculate the space-time metric tensor $g_{\mu\nu}$ and the motion of particles at the same time? In general relativity, we can't do it, the main reason is that the equation of motion, which is a corollary of the field equation, is not independent. That is, in general relativity, we have only one independent field equation, and it is impossible to solve $g_{\mu\nu}$ and $T_{\mu\nu}$ simultaneously. But the motion of microscopic particles essentially follows the laws of quantum mechanics, now we have a new wave equation(20) of particle motion, which is a basic equation independent of Einstein field equation. The new equation of motion(20) combined with Einstein field equation, together with the statistical interpretation of the wave function, can form a set of equations to solve $g_{\mu\nu}$ and $T_{\mu\nu}$ simultaneously.

Let the wave function of particle motion be

$$
\psi = A\mathrm{e}^{\frac{iS}{l_P}},
\tag{57}
$$

where A and S satisfy the wave equation

$$
\frac{\partial A}{\partial \tau} = -\frac{1}{2}\left(A\Box^2 S + 2\Box A \cdot \Box S + \frac{2A}{\sqrt{-g}}\Box\sqrt{-g}\cdot\Box S\right),
\tag{58}
$$

$$
\frac{\partial S}{\partial \tau} = -\frac{1}{2}(\Box S)^2 + \frac{1}{2} + \frac{l_P^2}{2}\frac{\Box^2 A}{A} + \frac{l_P^2}{2\sqrt{-g}}\Box^2\sqrt{-g} + \frac{l_P^2}{\sqrt{-g}A}\Box\sqrt{-g}\cdot\Box A,
\tag{59}
$$

then the energy momentum tensor of particle can be expressed as

$$
\begin{aligned}
T^{\mu\nu} &= \rho U^\mu U^\nu \\
&= A^2(\Box S)^\mu(\Box S)^\nu.
\end{aligned}
\tag{60}
$$

When the wave function of a particle ψ changes, the energy momentum tensor $T^{\mu\nu}$ also changes according to equation(60). It is assumed that Einstein field equation is still valid during dynamic change, that is, if $T^{\mu\nu}$ changes at a certain time and space point, the metric tensor at the same time and space point will change immediately to ensure that the field equation is still valid. Substituting formula(60) into Einstein field equation, and combining motion equations(58) and (59), we obtain the simultaneous

equations

$$\begin{cases} R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R=8\pi GA^2(\Box S)^\mu(\Box S)^\nu \\ \qquad\frac{\partial A}{\partial\tau}=-\frac{1}{2}\left(A\Box^2S+2\Box A\cdot\Box S+\frac{2A}{\sqrt{-g}}\Box\sqrt{-g}\cdot\Box S\right) \\ \qquad\frac{\partial S}{\partial\tau}=-\frac{1}{2}(\Box S)^2+\frac{1}{2}+\frac{l_P^2}{2}\frac{\Box^2A}{A}+\frac{l_P^2}{2\sqrt{-g}}\Box^2\sqrt{-g}+\frac{l_P^2}{\sqrt{-g}A}\Box\sqrt{-g}\cdot\Box A \end{cases} \tag{61}$$

where $R^{\mu\nu}$ is Ricci tensor, and the curvature scalar $R\equiv g^{\mu\nu}R_{\mu\nu}=R^\mu_\mu$. The wave function of particle ψ(i. e., A and S) and the metric tensor $g_{\mu\nu}$ of space-time can be obtained simultaneously by solving the above simultaneous equations. The solution of the equations(61) is the real state of particle motion, that is, a state of equilibrium between the motion of particle and the space-time structure.

(1). There are 12 unknown functions in the equation system(61): 10 metric tensors $g_{\mu\nu}$, wave function's amplitude A and its phase S. When the identity of curved space-time

$$\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)_{;\mu}=0 \tag{62}$$

is still valid, there are only 8 independent equations. The equations are underdetermined and can not completely determine $g_{\mu\nu}$, A and S. According to the common method of general relativity, 4 conditions about $g_{\mu\nu}$ can be added to form a complete set of equations together with 8 independent equations. These additional conditions are called gauge conditions or coordinate conditions. For example, a particularly convenient coordinate conditions are the harmonic coordinate conditions

$$\Gamma^\lambda\equiv g^{\mu\nu}\Gamma^\lambda_{\mu\nu}=0. \tag{63}$$

(2). The new wave equation(20) contains a constant Planck length l_P, when l_P tends to zero, the wave equation returns to the classical equation of motion of special relativity, but within the Planck scale, l_P cannot be ignored. Therefore, it is reasonable to believe that the new wave equation(20) is suitable for describing the motion of matter within the Planck scale. It should be a universal wave equation for particle motion in any space-time scale. If it is considered that the motion of matter always affects the corresponding changes of the space-time metric tensor through the Einstein field equation, the simultaneous equations(61) are still valid within the Planck scale. Within the Planck scale, the quantum fluctuations of space-time may make the identity(62) of curved space-time no longer hold, so the simultaneous equations(61) have 12 independent equations, 12 unknown functions, and the equations are complete. Therefore, the solution of the simultaneous equations(61) may open a window for us to understand the physics within the Planck scale, which needs further researches.

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